Cellular Automata on Cayley Graphs(ケイリーグラフ上のセルオートマタ)

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論文内容要旨

1 Background and Objectives

A cellular automaton is a network of identical, uniformly interconnected and synchronously clocked finite state machines. These interconnected machines are called cells, each of which interacts with its neighboring cells according to a uniform local interaction rule. Cellular automata provide simple and powerful models for parallel computation and natural phenomena. Researchers from computer science, biology, physics, and other fields, have been investigated cellular automata from various viewpoints. Many interesting results, both theoretical and experimental, have been already obtained and published.

The network topology of a cellular automaton is usually assumed to be a lattice in Euclidean n-space since most applications in the above mentioned fields fit in this setting. However, when we study crystal growth or physical phenomena in a curved space, we are naturally tempted to consider cellular automata with other network topologies such as fractals and Cayley graphs, which we will call non-Euclidean cellular automata. At present, attractive applications are rare, which confines the study of non-Euclidean cellular automata to a limited circle of theoretical researchers.

Even in this situation, if any of the rich classical Euclidean results is reproduced in the non-Euclidean framework with a proper reformulation, non-Euclidean cellular automata will attract more attention. This thesis focuses on non-Euclidean cellular automata and provides new methods of extending various classical results to non-Euclidean cellular automata and clarifies the difference between the Euclidean and the non-Euclidean cases.

The rest of this abstract is organized as follows. Section 2 gives basic definitions. Sections 3–6 describe the author's four major contributions to the non-Euclidean cellular automata theory. Concluding remarks and references are given at the end of this abstract.

2 Definitions and Notation

Let G be a finitely generated group. Let N be a finite subset of G that generates G. Let Q be a finite set called the set of *states*. A *local map* with *support* N is a map $\sigma : Q^N \to Q$. A map $x : G \to Q$ is called a *configuration*. Let C denote the set of all configurations, that is, Q^G. The *shift* s_g induced by $g \in G$ is a map $C \to C$ such that for any $x \in C$,

$$[s_g(x)](h) = x(g^{-1}h) \quad \text{for all } h \in G.$$

The *parallel map* T_σ induced by σ and N is a map $C \to C$ such that

$$(T_\sigma(x))(g) = \sigma(s_g^{-1}(x)|_N) \quad \text{for all } x \in C,\ g \in G,$$

where $s_g^{-1}(x)|_N$ denotes the restriction of $s_g^{-1}(x)$ to N. The pair (C, T_σ) is a discrete dynamical system and is called a *cellular automaton*.

We sometimes assume the existence of the *quiescent state* 0, that is, $\sigma(0, \ldots, 0) = 0$. The *support* of $x \in C$ is the set of all $g \in G$ with $x(g) \neq 0$, and denoted by $\mathrm{supp}(x)$. If $|\mathrm{supp}(x)| < \infty$, x is called a *finite configuration*, where $|A|$ denotes the number of elements of a set A. The set of all finite configurations is denoted by C_F. We denote by $\hat{T}_\sigma$ the restriction $T_\sigma|_{C_F} : C_F \to C_F$.

A group G is said to be *residually finite* if for any $g \in G$, there is a normal subgroup of finite index which does not contain g. A group G is said to have the *GOE* property if surjectivity of any parallel map T_σ is equivalent to injectivity of $\hat{T}_\sigma$.

3 Moore-Myhill psuedo-tilings

The *Garden-of-Eden*(GOE) theorem first appeared in Moore's paper. It claims that the existence of mutually erasable patterns implies the existence of a GOE pattern. A GOE pattern is a local configuration which cannot be reproduced in any environments. If a configuration contains a copy of a GOE pattern, the configuration cannot be reproduced. Therefore a *self-reproducing* configuration must not contain any copy of a GOE pattern.

In 1993, Machì and Mignosi proved the Garden of Eden (GOE) theorem for cellular automata on Cayley graphs of non-exponential growth. This was the first nontrivial non-Euclidean result that corresponds to fundamental results in Euclidean cellular automata theory. The extended GOE theorem was a great preliminary step in the sense that, since many important classical results rely on the GOE theorem, many of non-Euclidean extensions rely on the extended GOE theorem.

Aside from Machì and Mignosi' success, the author took a different approach in which the cellular spaces are restricted to a particular class of groups and found that a Moore-Myhill like tiling is still possible for them[B]. Compared to Machì and Mignosi's general result, the proof has the advantage that it fully exploits the particular group structure so that a Moore-Myhill like tiling can be explicitly constructed. The tiling, as the wording suggests, is a spatially economic covering. The nonuniform packing method is introduced as a key technique for the construction.

4 Dynamics of Cellular Automata on Groups

Let $\mathcal{A} = (Z^d, Q, N, \sigma)$ be a cellular automaton. A configuration x is said to be *periodic* or *cofinite* if it is spatially periodic with respect to all directions in Z^d. C_P denotes the set of all periodic configurations. A configuration $x \in C$ is said to be *Poisson stable* with respect to T_σ if $(T_\sigma)^n(x)$ returns infinitely many times nearby x. Let M be a subset of C. A parallel map T_σ is said to be *M Poisson stable* if every $x \in M$ is Poisson stable with respect to T_σ. A configuration $x \in C$ is said to be *strongly Poisson stable* with respect to T_σ if $(T_\sigma)^n(x) = x$ for infinitely many $n > 0$. A parallel map T_σ is said to be *M strongly Poisson stable* if every $x \in M$ is strongly Poisson stable with respect to T_σ.

Now we can state part of Sato and Honda's result (1977) which we will focus on.

Theorem (Sato and Honda). *The following five conditions are arranged in the order of strength, that is,* (i) *implies* (ii), (ii) *implies* (iii), *and so on. In* (iii), *all the subitems are equivalent conditions.*

(i) *injective on C.*

(ii) *period preserving on C.*

(iii) *(a) strongly C_P-Poisson stable*
(b) C_P-Poisson stable
(c) injective on C_P
(d) finite order on C
(e) finite order on C_F
(f) finite order on C_P

(iv) *surjective and period preserving on C_P*

(v) *surjective on C*

A non-Euclidean extension of this theorem is not straightforward since Euclidean theory uses the GOE theorem at crucial steps which does not hold in general for non-Euclidean cellular automata. Various periodic constructions in Sato and Honda's work turned out to be valid only when the underlying group has residual finiteness. The author showed the conditions on groups under which non-Euclidean extensions are possible [C]. This clarifies why classical methods worked well in the Euclidean case. Periodic constructions for non-Euclidean cellular automata are based on the notion of period that has been emerged from [A].

The following theorem is obtained as an extension in which the cell space Z^d is replaced by a group G.

Theorem (Yukita). *Let G be a finitely generated group.*

(i) *If T_σ is injective on C, then it is period preserving on C.*

(ii) *Let G has the GOE property and is residually finite. If T_σ is period preserving on C, then it is injective on C_P.*

(iii) *The following four conditions are equivalent.*

   (a) strongly C_P-Poisson stable

   (b) C_P-Poisson stable

   (c) injective on C_P

   (d) surjective and period preserving on C_P

(iv) *Let G be residually finite. If T_σ is surjective on C_P, then it is surjective on C.*

(v) *Let G has the GOE property and is residually finite. If T_σ is period preserving on C_P, then it is surjective on C.*

5 Cellular Automata with Asymptotic Boundary Conditions

Maruoka and Kimura introduced variants of the notions of injectivity and surjectivity — the notions of weak injectivity/surjectivity and strong injectivity/surjectivity — and obtained results concerning the hierarchy among those properties (1976, 1979, 1982), which we will call *Maruoka-Kimura's hierarchy*, or the *M-K hierarchy* for short. We will also use the same term to refer to the non-Euclidean extensions of the M-K hierarchy.

Two configurations are *asymptotically equivalent* if they take equal states for all but a finite number of cells. If any two configurations are asymptotically equivalent, so are their successors. Let C_x denote the set of all configurations that are asymptotically equivalent to x, and let $T_{\sigma,x}$ denote the restriction $T_\sigma|_{C_x} : C_x \to C_{T_\sigma(x)}$. A parallel map T_σ is *weakly injective* if $T_{\sigma,x}$ is injective for some $x \in C$, and *strongly injective* if $T_{\sigma,x}$ is injective for all $x \in C$. A parallel map T_σ is *weakly surjective* if $T_{\sigma,x}$ is surjective for some $x \in C$, and *strongly surjective* if $T_{\sigma,x}$ is surjective for all $x \in C$. A parallel map T_σ is *residually injective* if no two asymptotically non-equivalent configurations have asymptotically equivalent successors. A parallel map T_σ is *residually surjective* if any configuration x has an asymptotically equivalent configuration that has a predecessor. The terms *totally injective/surjective* are meant for surjectivity and injectivity on C.

Maruoka and Kimura's result shows that relations among properties of injectivity, surjectivity, their strong and weak versions form a hierarchical structure:

Theorem (Maruoka and Kimura). *Let $G = Z^d$. In each of the following* (i) *and* (iii), *all the conditions are equivalent. Further, conditions* (i) *implies* (ii), *and* (ii) *implies* (iii).

(i) *(a) residually injective*

   (b) totally injective

   (c) strongly surjective

(ii) *weakly surjective*

(iii) *(a) strongly injective*

(b) weakly injective

(c) totally surjective

(d) residually surjective

Attempts at non-Euclidean extensions must face the following problem. Maruoka and Kimura's discussions depend heavily on the notions of *balancedness* and *hardness* and the following facts. Surjectivity and injectivity of parallel maps are characterized as:

1. A parallel map is surjective if and only if it is balanced.

2. A parallel map is injective if and only if it is hard.

Neither balancedness, hardness, nor these characterizations work well for non-Euclidean cellular automata. Therefore, the author had to seek other approaches for a non-Euclidean extension and eventually obtained several versions of modified hierarchies in [D], where various conditions are imposed in turn on the groups that generate the tessellation. The conditions considered were the *GOE property*, *residual finiteness*, and their combination.

The following theorem is obtained as an extension in which the cell space Z^d is replaced by a group G, where the condition on groups is taken the most general.

Theorem (Yukita). *Let G be a finitely generated infinite group.*

(i) *If T_σ is totally injective, then it is strongly(weakly) injective.*

(ii) *T_σ is strongly injective if and only if it is weakly injective.*

(iii) *If T_σ is strongly surjective, then it is weakly surjective.*

(iv) *If T_σ is weakly surjective, then it is totally surjective.*

(v) *If T_σ is totally surjective, then it is residually surjective.*

The most strict condition on groups considered is "GOE + Residual Finiteness." Under this condition we can restore nearly all of the M-K hierarchy.

6 Linear Cellular Automata on Groups

One branch of cellular automata theory concentrates on linear local rules. Linearity provides rich computable structures and enables us to obtain nontranscendental results that have applications in many areas.

Let $Z_m = \{\bar{0}, \ldots, \overline{m-1}\}$ be the ring of residue classes modulo m. Consider a cellular automaton $\mathcal{A} = (Z^d, Q, N, \sigma)$, where $Q = Z_m$ and σ is given by

$$\sigma(u_1, \ldots, u_n) \equiv a_1 u_1 + \cdots + a_n u_n \pmod{m},$$

with $a_i \in Z$ for $1 \leq i \leq n$. Such a cellular automaton with a linear local rule is called a *linear cellular automaton.* Let S denote the set of all coefficients a_i appearing in the specification of σ. Let Spec $(m) = \{p_1, \ldots, p_s\}$ be the set of all prime factors of m. This set is partitioned as Spec $(m) = W_{m,S} \cup P^{(1)}_{m,S} \cup \cdots \cup P^{(n)}_{m,S} \cup Q_{m,S}$, where each set is determined as follows. $W_{m,S}$ is the set of prime factors of m that divide all of $a_1, \ldots, a_n$. $P^{(i)}_{m,S}$ is the set of prime factors of m that do not divide a_i but divide all other coefficients. $Q_{m,S}$ is the set of prime factors of m that do not divide at least two coefficients a_i, a_j $(i \neq j)$.

Ito, Osato, and Nasu obtained the following two theorems that claim that injectivity and surjectivity of parallel maps of linear cellular automata are completely determined by the corresponding local rules. Further studies on this track were conducted by Aso and Honda(1985) and recently by Manzini and Margara(1997,1999).

Theorem (Ito, Osato, and Nasu). *The following three properties are equivalent.*

(i) $\hat{T}_\sigma$ *is injective.*

(ii) T_σ *is surjective.*

(iii) $W_{m,S} = \emptyset$.

Theorem (Ito, Osato, and Nasu). *The following three properties are equivalent.*

(i) T_σ *is injective.*

(ii) $\hat{T}_\sigma$ *is surjective.*

(iii) $W_{m,S} = Q_{m,S} = \emptyset$.

A non-Euclidean extension must face the serious difficulty caused by the absence of commutativity. Ito-Osato-Nasu and Aso-Honda's arguments heavily depend on the algebraic nature of the group Z^d or Abelian groups. The author examined how an attempt of non-Euclidean extension fails for various groups and obtained a sufficient condition on groups that allows Ito-Osato-Nasu type theorems [E]. The above result on injectivity and surjectivity is derived as a corollary of the author's result. In addition, the techniques used in the proofs clarify the algebraic nature of Ito-Osato-Nasu's theorems, which was only implicitly described in their paper. The proofs utilize properties of unique product groups and Machì and Mignosi's GOE theorem.

The following two theorems are obtained as extensions. The cell space Z^d is replaced by a group G.

Theorem (Yukita). *Let G be a unique product group with a finite set of generators N and have the GOE property. The following three properties are equivalent.*

(i) $\hat{T}_\sigma$ *is injective.*

(ii) T_σ *is surjective.*

(iii) $W_{m,S} = \emptyset$.

Theorem (Yukita). *Let G be a unique product group with a finite set of generators N and have the GOE property. The following three properties are equivalent.*

(i) T_σ *is injective.*

(ii) $\hat{T}_\sigma$ *is surjective.*

(iii) $W_{m,S} = Q_{m,S} = \emptyset$.

7 Conclusions

Non-Euclidean extensions of GOE, dynamical properties, asymptotic theory, linear theory are obtained. Differences between Euclidean and non-Euclidean are characterized by the properties of groups: GOE, residual finiteness, and the unique product property. Linear cellular automata without the unique product property provide new concrete examples of genuinely non-Euclidean nature in a systematic manner.

Main Publications related to the Theme

[A] S. Yukita. Tessellation automata on free groups. *Hiroshima Math. J.*, 25(3):561–570, 1995.

[B] S. Yukita. The Moore-Myhill pseudo tiling for the Heisenberg tessellation automata. *Japan J. Indust. Appl. Math.*, 16(1):47–63, 1999.

[C] S. Yukita. Dynamics of cellular automata on groups. *IEICE Trans. Inf. & Syst.*, E82-D(10):1316–1323, 1999.

[D] S. Yukita. Cellular automata on groups with asymptotic boundary conditions. *Trans. Inform. Processing Soc. Japan*, 40(12):4151-4158, 1999.

[E] S. Yukita. Linear cellular automata on Cayley graphs. *Japan J. Indust. Appl. Math.*, (to appear), 2000.

論文審査の結果の要旨

セルオートマトンは，セルと呼ばれる有限オートマトンを一様に接続した計算モデルであるが，自己増殖を始めとして，様々の現象を模倣するために用いられてきた．このモデルを，広範囲の現象を説明する普遍的なものとするためには，モデルの一般化やセルオートマトンの動作の解明など，解決すべき課題は多い．本論文は，セル間の接続を格子状とする従来のモデルを一般化した，群に基づいて決まる接続を持つモデルを取りあげ，並列写像や様相の時系列の振る舞いを，セル間の接続を定める群に関する諸条件との関連を示しながら解明したもので，全編7章よりなる．

第1章は序論である．

第2章では，セルの間の接続を群の要素を乗ずるという関係に基づくケイリーグラフに一般化するとともに，本論文を通して用いられるセルオートマトンの定義を与えている．

エデンの園の定理は，消滅パターンが存在すれば，初期設定によってしか出現し得ないエデンの園パターンが存在することを主張するものである．第3章では，この定理がセル間の接続をハイゼンベルク群に基づいたケイリーグラフに一般化しても成立することを導いている．この定理の証明では，消滅パターンからエデンの園パターンを具体的に構成しており，これは優れた成果である．

第4章では，並列写像が単射であれば，空間周期を保存すること，また，並列写像が周期様相上で単射であることと，周期様相に関してポアソン安定となることが同値となること等を導いている．さらに，セル間接続を定める群が剰余有限性を有し，エデンの園の定理を成立させるという条件を満たす場合は，並列写像が様相の空間周期を保存するならば，周期様相上で単射となることを導いている．これは興味深い成果である．

第5章では，強全射ならば弱全射となることと弱単射と強単射は同値となることを導くとともに，セル間接続を定める群がエデンの園の定理を成立させるという条件を満たす場合は，弱全射ならば弱単射となること等を示している．

第6章では，線形セルオートマトンを取りあげ，セルオートマトンのセル間接続を定める群に積分解の一意性を仮定して，並列写像が単射，あるいは全射となる条件を，それぞれ局所写像の係数に関する数論的な条件として与えている．

第7章は結論である．

以上要するに本論文は，セル間接続を格子型から群に基づいた接続に一般化し，セルオートマトンが実現する並列写像と，並列写像の繰り返し適用により得られる様相の時系列を対象として，単射性，全射性，周期性等に関する多くの有用な知見を導いたもので，情報基礎科学の発展に寄与するところが少なくない．

よって，本論文は博士(情報科学)の学位論文として合格と認める．